Harmonic Motion

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Content of the course

pendulum

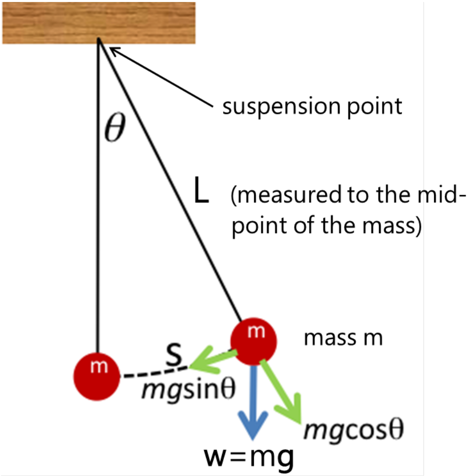

Figure: Simple harmonic motion

For small θ

$$F_\theta = -mg\sin\theta \approx -mg\theta \tag{1}$$

So the equation of motion

$$\ddot{\theta} = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\theta \tag{2}$$

or

$$\begin{aligned}\frac{\mathrm{d}\omega}{\mathrm{d}t} &= -\frac{g}{l}\theta \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \omega\,.\end{aligned}$$

Euler method solution

$$\begin{aligned}\theta_{i+1} &= \theta_i + \omega_i\Delta t \\ \omega_{i+1} &= \omega_i - \frac{g}{l}\theta_i\Delta t\,,\end{aligned}$$

where $\theta_i = \theta(t_i)\,,\ \omega_i = \omega(t_i)$

Euler methods fails

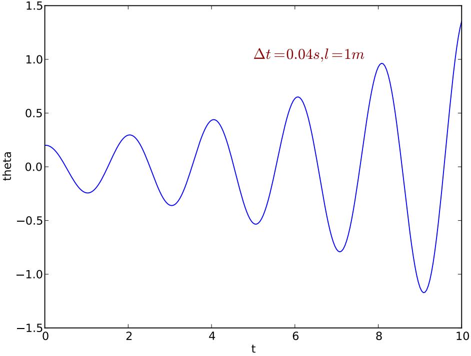

Figure: Euler method to calculate theta

What happen if we choose smaller Δt ?

Euler method is unstable for harmonic motions

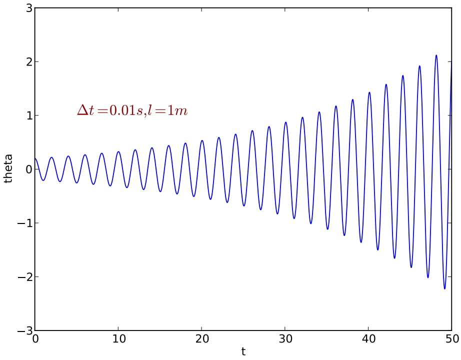

Figure: Euler method to calculate theta

what is wrong with Euler method?

Let us look at the total energy:

$$E = E_{\rm kin} + E_{\rm pot} \; = \; \frac{m}{2} l^2 \omega^2(t) + mgl[1 - \cos\theta(t)] \approx \frac{m}{2} l^2 \omega^2(t) + \frac{m}{2} gl\theta^2(t)$$

The total enery in the next step:

$$\begin{aligned} E_{i+1} &= \frac{ml^2}{2}\left[\omega_{i+1}^2 + \frac{g}{l}\theta_{i+1}^2\right] = \frac{ml^2}{2}\left[\left(\omega_i - \frac{g}{l}\theta_i \Delta t\right)^2 + \frac{g}{l}(\theta_i + \omega_i \Delta t)^2\right] \\ &= \frac{ml^2}{2}\left[\omega_i^2 + \frac{g}{l}\theta_i^2\right] + \frac{mgl}{2}\left(\frac{g}{l}\theta_i^2 + \omega_i^2\right)(\Delta t)^2 \\ &= E_i + \frac{mgl}{2}\left(\frac{g}{l}\theta_i^2 + \omega_i^2\right)(\Delta t)^2 \; . \end{aligned}$$

So the total energy is not conserved!

Euler-Cromer algorithm

Instead of Euler method:

$$\begin{aligned}\theta_{i+1} &= \theta_i + \omega_i \Delta t \\ \omega_{i+1} &= \omega_i - \frac{g}{l}\theta_i \Delta t\,,\end{aligned}$$

We do a small change, (Euler-Cromer algorithm):

$$\begin{aligned}\omega_{i+1} &= \omega_i - \frac{g}{l}\theta_i \Delta t \\ \theta_{i+1} &= \theta_i + \omega_{i+1} \Delta t\,.\end{aligned}$$

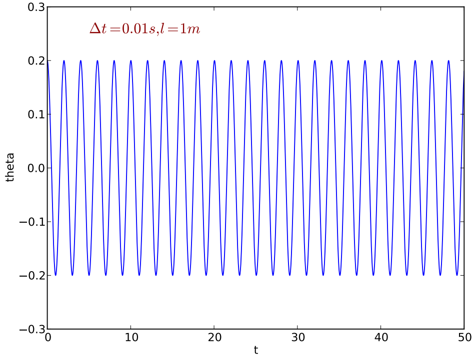

Figure: Euler-Cromer method to calculate theta

The Euler-Cromer method conserves energy over each complete period of the motion.

A more general problem

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\sin\theta \tag{3}$$

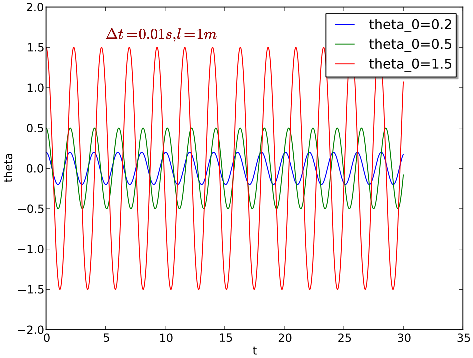

Figure: Euler-Cromer method to calculate theta

A more general problem

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\sin\theta - q\frac{\mathrm{d}\theta}{\mathrm{d}t} + F_D \sin(\Omega_D t) \ . \tag{4}$$

- Where $-q\frac{\mathrm{d}\theta}{\mathrm{d}t}$ is damping (here q is a parameter that is a measure of the strength of the damping),
- $F_D \sin(\Omega_D t)$ is a sinusoidal driving force with amplitude F_D and angular frequency Ω_D

Example 1

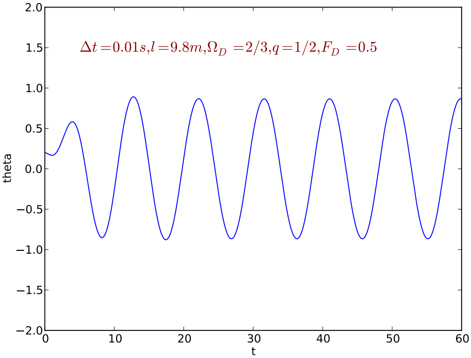

Figure: $F_D = 0.5, \; q = 1/2, \; \Omega_D = 2/3, \; \theta_0 = 0.2$

Example 2

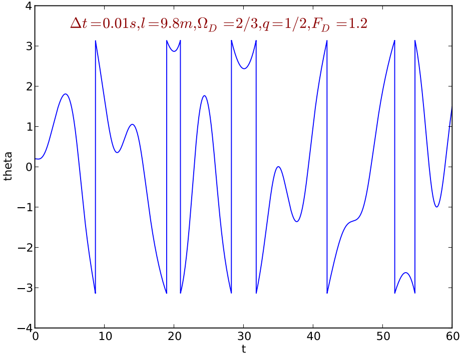

Figure: $F_D = 1.2,\ q = 1/2,\ \Omega_D = 2/3,\ \theta_0 = 0.2$

The vertical jumps in θ occur when the angle is reset so as to keep it in the range $-\pi$ to $+\pi$; they do not correspond to discontinuities in $\theta(t)$

Part of the python program

```
for i in range(n):

    omega[i+1]=omega[i] - g/l *sin(theta[i])*dt \
               -q * omega[i]*dt + F_D*sin(omega_D*t[i])*dt
    theta[i+1]=theta[i] + omega[i+1]*dt
    t[i+1] = t[i] +dt
    if theta[i+1] >  pi:
       theta[i+1] = theta[i+1] - 2.0 * pi
    if theta[i+1] <-pi:
       theta[i+1] = theta[i+1] + 2.0 * pi
```

chaos in the harmonic motions

- Imagine two identical pendulums, with the same lengths, damping factors and driving forces. The only difference is initial angles(θ_0).
- for first pendulum, we can calculate the angular positions of pendulum,$\theta_1(t)$ and for second $\theta_2(t)$
- we can calculate $\Delta\theta(t) = |\theta_1(t) - \theta_2(t)|$

no chaos

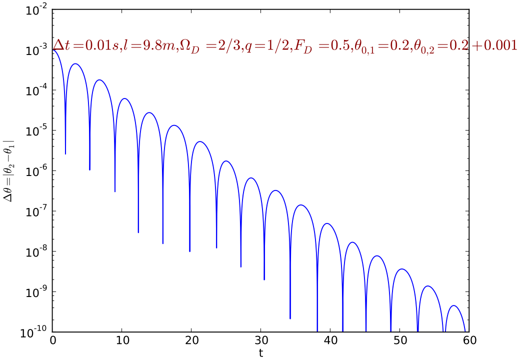

Figure: $F_D = 0.5,\ q = 1/2,\ \Omega_D = 2/3,\ \theta_{0,1} = 0.2, \theta_{0,2} = 0.2 + 0.001$

chaotic regime

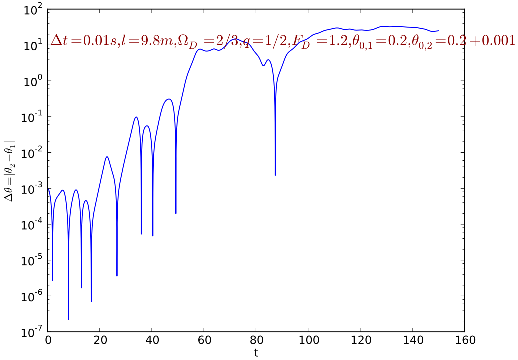

Figure: $F_D = 1.2,\ q = 1/2,\ \Omega_D = 2/3,\ \theta_{0,1} = 0.2, \theta_{0,2} = 0.2 + 0.001$

Lyapunov exponent

From two previous plots, we find a overall behavior:

$$\log(\Delta\theta) \sim \lambda t \tag{5}$$

Or:

$$\Delta\theta \approx e^{\lambda t} \tag{6}$$

The parameter λ is known as a Lyapunov exponent
The behavior of $\Delta\theta$ can be described by a Lyapunov exponent in both the chaotic and nonchaotic regimes. In the former case $\lambda > 0$, while in the latter, $\lambda < 0$. The transistion to chaos thus occures when $\lambda = 0$

phase-space, nonchaotic

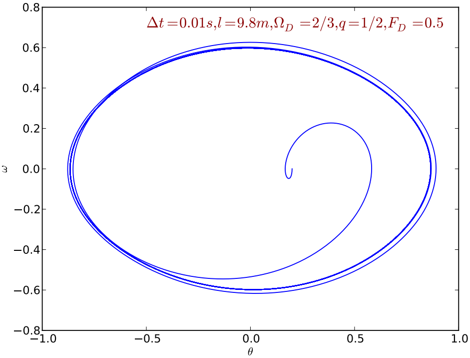

Figure: $F_D = 0.5,\ q = 1/2,\ \Omega_D = 2/3,\ \theta_0 = 0.2$

phase-space, chaotic

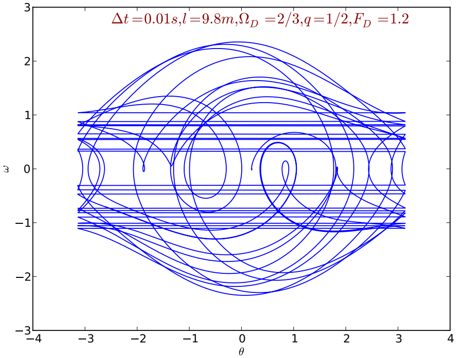

Figure: $F_D = 1.2,\ q = 1/2,\ \Omega_D = 2/3,\ \theta_0 = 0.2$

exercise 3

- show that Euler method does not work for a pendulum but Euler-Cromer does (plot θ vs. time with two methods)
- plot total energy vs. time for Euler and Euler-Cromer method for a pendulum
- plot $\log(\Delta\theta(t)) = |\theta_2(t) - \theta_1(t)|$ for $F_D = 0.5$ and $F_D = 1.2$. Suppose: $q = 1/2,\ l = 9.8\ g = 9.8\ \Omega_D = 2/3,\ \theta_{0,1} = 0.2,\ \theta_{0,2} = \theta_{0,1} + 0.001$