Laplace's Equation

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Content of the course

Electric potential

For regions of space that do not contaion any charges:

$$\nabla^2 V(x,y,z) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{1}$$

For an arbitrary boundary condition we have to use a numerical method.

Numerical solution of $\nabla^2 V = 0$ in two dimensions

We divide the space to small segments; Δx and Δy. By using Taylor expantion:

$$
\begin{aligned}
V(x+\Delta x, y) &= V(x,y) + \Delta x \frac{\partial V(x,y)}{\partial x} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 V(x,y)}{\partial x^2} \\
V(x, y+\Delta y) &= V(x,y) + \Delta y \frac{\partial V(x,y)}{\partial y} + \frac{1}{2}(\Delta y)^2 \frac{\partial^2 V(x,y)}{\partial y^2} \\
V(x-\Delta x, y) &= V(x,y) - \Delta x \frac{\partial V(x,y)}{\partial x} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 V(x,y)}{\partial x^2} \\
V(x, y-\Delta y) &= V(x,y) - \Delta y \frac{\partial V(x,y)}{\partial y} + \frac{1}{2}(\Delta y)^2 \frac{\partial^2 V(x,y)}{\partial y^2}
\end{aligned} \quad (2)
$$

Sum of these equations gives the following equation (we suppose $\Delta x = \Delta y$ and we know that $\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0$, Laplace's equation):

$$V(x+\Delta x, y) + V(x, y+\Delta y) + V(x-\Delta x, y) + V(x, y-\Delta y) = 4V(x,y) \quad (3)$$

Therefore:

$$V(x,y) = \frac{1}{4}[V(x+\Delta x, y)+V(x-\Delta x, y)+V(x, y+\Delta y)+V(x, y-\Delta y)] \tag{4}$$

And for three-dimensions

$$\begin{aligned} V(x,y,z) = \; & \tfrac{1}{6} \; [V(x+\Delta x, y, z) + V(x-\Delta x, y, z) \\ + \; & V(x, y+\Delta y, z) + V(x, y-\Delta y, z) \\ + \; & V(x, y, z+\Delta z) + V(x, y, z-\Delta z)] \end{aligned} \tag{5}$$

We fix the values of $V(x,y,z)$ at boundary and in an iterative process we recalculate $V(x,y,z)$ from above equation until our result satisfies some convergence criteria.

relaxtion method

$$V_n(x,y) \xrightarrow{\text{update by equation (3)}} V_{n+1}(x,y) \quad (6)$$

Where n index shows the iteration number i.e. $V_0(x,y)$ are intial values for $V(x,y)$ (we set for example $V_0(x,y)=0$), $V_1(x,y)$ are values which optained from equation (4) by putting $V_0(x,y)$ in the right side of equation, and so on or:

$$V_{n+1}(x,y) = \frac{1}{4}[V_n(x+\Delta x,y)+V_n(x-\Delta x,y)+V_n(x,y+\Delta y)+V_n(x,y-\Delta y)] \quad (7)$$

Or in more simple way:

$$\begin{aligned} V_{new}(x,y) = \quad & \frac{1}{4} \quad [V_{old}(x+\Delta x,y) + V_{old}(x-\Delta x,y) \\ & + \quad V_{old}(x,y+\Delta y) + V_{old}(x,y-\Delta y)] \end{aligned}$$

We stop iterations when (convergence criteria):

$$|V_{new} - V_{old}| < \epsilon \quad (8)$$

Where ϵ is a small number(The smaller ϵ means the more accuracy).

An example

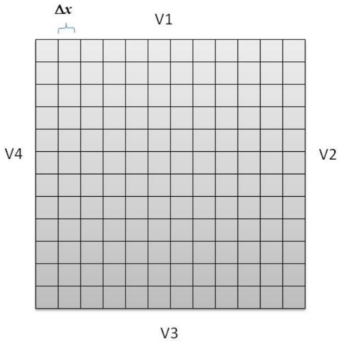

Figure: A square with boundary conditions at its edges, V_1, V_2, V_3, V_4

An example

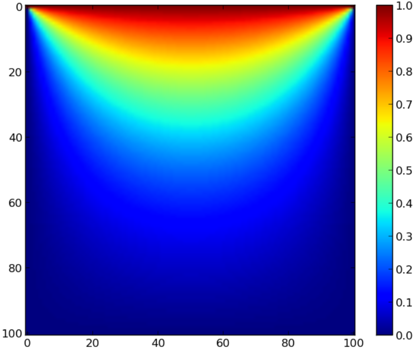

Figure: Potential inside a square with boundary conditions at its edges: $V_1 = 1,\ V_2 = 0,\ V_3 = 0,\ V_4 = 0$

An example

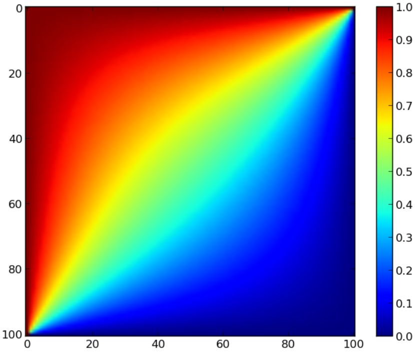

Figure: Potential inside a square with boundary conditions at its edges: $V_1 = 1,\ V_2 = 0,\ V_3 = 0,\ V_4 = 1$

exercise 4, part 1

- Suppose a squre with different edge potentails (V_1, V_2, V_3, V_4)
- plot a counter plot for potential inside the squre

hint

```
from numpy import empty, zeros, max
from pylab import imshow, gray, show, colorbar
N = 100   # Grid squares on a side
eps= 1e-5 #  accuracy
V_old = zeros([N+1, N+1], float)  # Create arrays to hold potential va
#some part is missing ?
V_new = empty( [N+1,N+1], float)
diff = 1.0
while diff > eps:  # Main loop
    # Calculate new values of potential
    for i in range(N+1):
        for j in range(N+1):
    # some part is missing
    # Claculate maximum difference from old values
    diff = max(abs(V_old-V_new))
    # Swap the two arrays around
    V_old, V_new = V_new, V_old
imshow(V_old) # Make a plot
colorbar()
show()
```

Solution of the poisson equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{9}$$

From equation (2) we have:
(we suppose $\Delta x = \Delta y$ and we know that $\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = -\frac{\rho}{\epsilon_0}$, Poissin equation):

$$V(x+\Delta x, y)+V(x, y+\Delta y)+V(x-\Delta x, y)+V(x, y-\Delta y) = 4V(x, y)-\frac{\rho(x, y)}{\epsilon_0}(\Delta x)^2 \tag{10}$$

So

$$V(x, y) = \frac{1}{4}[V(x+\Delta x, y)+V(x-\Delta x, y)+V(x, y+\Delta y)+V(x, y-\Delta y)]+\frac{(\Delta x)^2}{4\epsilon_0}\rho(x, y) \tag{11}$$

exercise 4, part2

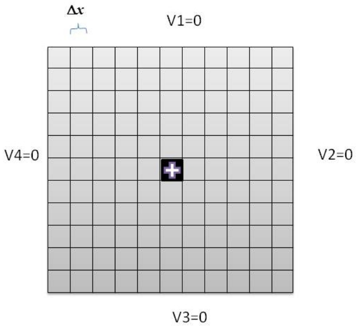

Figure: A square with boundary conditions at its edges, $V_1 = 0$, $V_2 = 0$, $V_3 = 0$, $V_4 = 0$ and a point charge inside

exercise 4, part2

- Suppose a squre with edge potentails, $V = 0$ and a point charge (assume $q/\epsilon_0 = 1$)at the center.
- plot a counter plot for potential inside the squre